$\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}$.

$C(\mathbb{T}) := \{$ all continuous functions defined on $\mathbb{T} \}$.

$\int_{\mathbb{T}} f(z)\,dz := \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\,dt$.

An inner product on $C(\mathbb{T})$ is given by $(f, g) = \int_{\mathbb{T}} f(z)\overline{g(z)}\,dz$.

$L^2(\mathbb{T})$ is the completion of $C(\mathbb{T})$ under the norm $\|\cdot\|_2$ induced by the inner product.

$f_n(z) = z^n$.

Proposition

$\{f_n : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$

Proof: (i) Orthogonal:

For any $m, n \in \mathbb{Z}$ with $m \neq n$,

$(f_n, f_m) = \int_{\mathbb{T}} f_n(z)\overline{f_m(z)}\,dz$

$$= \int_{\mathbb{T}} z^n \bar{z}^m dz$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)t} dt$$

$$= \frac{1}{2\pi(n-m)} e^{i(n-m)t}\Big|_0^{2\pi}$$

$$= 0$$

(ii) Normal:

For any $n \in \mathbb{Z}$,

$$(f_n, f_n) = \int_{\mathbb{T}} z^n \bar{z}^n dz$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-n)t} dt$$

$$= \underline{1} \ .$$

(iii) Maximal:

By Proposition 14.8, it suffices to show if $(g, f_n) = 0$ for all n, then $g = 0$.

$$\boxed{\begin{array}{l}\text{Stone-Weierstrass Theorem:} \\ \text{Every element in } L^2(\mathbb{T}) \text{ can be approximated} \\ \text{by polynomials with respect to } \|\cdot\|_2\end{array}}$$

For any $g \in L^2(\mathbb{T})$, there exists a sequence (P_m) of polynomials such that $\|g - P_m\|_2 \to 0$ as $m \to \infty$.

Suppose $(g, f_n) = 0$ for all $n \in \mathbb{Z}$.

Then $(g, P_m) = 0$ for all m.

$$\|g - P_m\|_2^2 = (g - P_m, g - P_m) = \|g\|_2^2 + \|P_m\|_2^2.$$

Then $\|g\|_2^2 \le \lim_{m \to \infty} \|g - P_m\|_2^2 = 0$.

Hence, $g = 0$.

$\square$

Recall

Corollary 14.12

Let X be a separated Hilbert space.

If $\dim X = n$, then $X = \mathbb{C}^n$.

If $\dim X = \infty$, then $X = \ell^2$.

Notice that $\{$all the polynomials$\}$ is countable.

Again, by Stone-Weierstrass Theorem, $L^2(\mathbb{T})$ is a separate Hilbert space. Since $\dim L^2(\mathbb{T}) = \infty$, by Corollary 14.12, $L^2(\mathbb{T}) \cong \ell^2$.